State of the Collatz-Conjecture

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Abstract—**We discuss a claimed proof to the Collatz-Conjecture that has been published by Peter Schorer. This is, we explain the basic concepts, terminology and lemmas he uses, as well as the proof itself. After reading this paper the reader should be able to quickly orient herself in Schorer's original paper. Secondly we discuss and name several points in his work that give reason to critisism and skepticism. A short Haskell-program has been developed along with this paper and can be found in the appendix.**

Index Terms—**Collatz-Conjecture, 3x+1 Problem, Proof, Peter Schorer, Tuple-Set**

I. Introduction

In February 2009, Peter Schorer published the latest version of his paper *A Solution to the* $3x + 1$ *Problem* [**?**] in which he claims a proof of the Collatz-conjecture. For more than 70 years not only mathematicians all over the world tried to solve this problem [**?**]. This gives us reason to have a closer look at his claim.

We try to give an insight to his paper in two ways. Firstly we explain the basic concepts used, as well as the important lemmas and of course, the idea behind the proof he gives. Secondly, we will discuss points of criticism and the possibility of flaws.
Because of the limited length of our paper, we will not be able to present every lemma and proof here. But we are using the same terminology that is used in Peter Schorer's paper so that the interested reader should be able to quickly orient herself there in order to find further information. We will also give references in our text to the corresponding pages in his paper where possible.

II. Basic Concepts

We would like to begin this section with a quote by J. C. Lagarias. He writes in the conclusion in one of his publications:

> Of course there remains the possibility that someone will find some hidden regularity in the 3x+1 problem that allows some of the conjectures about it to be settled. [**?**]

Our paper shows why Schorer might fulfill this statement. At least he approaches the problem with concepts that we have not seen elsewhere, yet. Although his concept of *exponent sequences* has been known before as the term *admissible vector* (for example in publications by Wirsching [**?**], [**?**] and Lagarias [?]), there is a difference in the use of it. On the other hand, a radical change in concepts involves a higher risk of mistakes and gives reason to criticism as we will see in the last section of this paper.

A. Recap of the Collatz-Conjecture

Before we begin with introducing the basic concepts, we would like to start by defining the conjecture. The most common way ([?])

of stating the Collatz-Conjecture is with help of this formula

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ 3x+1 & \text{if } x \text{ is odd} \end{cases} \quad (1)$$

We assume $x \in \mathbb{N}$. Then the range of f is $\mathbb{N}$ because we only divide by two if x is even and it does not include negative numbers because there is no subtraction in the definition. Therefore we can always apply f to its return value.

Starting with any value x, then, infinitely-often applications of f will result in an infinite sequence of elements whose values have no upper bound, or an infinite sequence of elements that contains an infinitely repeated cycle.

By testing, one can immediately find the cycle 1-4-2-1. In fact, every tested number so far [**?**] ends in this cycle and the Collatz-conjecture proposes that every number does. Accordingly, we call a number that ends in this cycle a *non-counterexample* and a *counterexample* otherwise [?, p. 6].

B. Exponent Sequences and Tuples

Underlying all following discussions are the concepts of *exponent sequences* and *tuples*. The ideas of these arise when we look at (1) in the following way:

$$C(x) = \frac{3x+1}{2^a},\ x \in \mathbb{N} \text{ and odd} \quad (2)$$

With $a \in \mathbb{N}$ such that $C(x) \in \mathbb{N}$ and odd. In other words, a is the number of times that we would have to apply the case $x/2$ of the former definition (1). Obviously there exists such an a for any odd $x \in \mathbb{N}$ because $3x+1$ is then even and therefore at least once divisible by two. Also, a is unique, since it is the maximum value such that C returns an integer value.

Similar to the conventional definition (1), we can apply this function an arbitrary number of times to its return value to receive a sequence of intermediate values which is then analyzed. However, with the new definition, the intermediate values between each steps will be only the odd values. This sounds like we are lacking information here but note that we also get a sequence of values of the exponent a with each step. These two sequences are subject of further analysis from now on. Following Schorer's terminology, we call the sequence of values of x *tuple* and the sequence of values of a *exponent sequence* [?, pp. 4].

Remark: Tuples starting with even numbers do not exist. The behavior of the Collatz-function on even numbers is reduced to its behavior on odd numbers. For reduction we use implicit repeated division by 2. This is possible because if there is any even counterexample, then and only then the odd number it is reduced to will also be a counterexample.

Let's look at three important features of tuples: As we reasoned before, for every odd positive integer x, there is a tuple starting with x. Also, since we can always apply C to its return value, all tuples are infinitely long. And clearly, every tuple is only defined by its start number.

C. Tuple-Sets and Levels

Now, let us have look at the basic relation between *exponent sequences* and *tuples*: Given a tuple, it defines precisely one exponent sequence. To see this, recall that in our definition (2) of C for every x there exists precisely one a. More interesting however, is the other way around: Given an exponent sequence, is it too, associated with only one tuple? For simplicity, we focus at *finite* exponent sequences (and therefore finite tuples) first. We call tuples that consist only of i elements i-level tuples and the exponent sequences they define i-level exponent sequences (note that i-level exponent sequences consist of $i-1$ elements).

Remark: The reader is advised to pay particularly attention to passages where no explicit level is given. In this case, we are still dealing with the infinite sequences!

To find the answer to our last question, let us look at an example: The 2-level exponent sequence consisting of only one element, $\{1\}$, is associated with many 2-level tuples, for example $< 3, 5 >$ because $(3 \cdot 3 + 1)/2^1 = 5$. Or similarly $< \ 51, 77 \ >$. So we have to answer our antecedent question with *No* but it brings us into the position that we can say that a finite exponent sequence defines a *set* of tuples. Then, which elements would be in such a set?

The obvious answer would be: It contains all the tuples that define the given exponent sequence. However, this is not the definition Schorer uses. He calls these sets *tuple-sets* and defines them as follows: A *tuple-set* defined by an exponent sequence A contains tuples that define the longest possible prefix of A [?, p.7].

Explicitly, the term *prefix* includes also the empty sequence and A itself. Let's also clarify what is meant by the term *longest possible prefix*. By this we exclude tuples that can be extended by another application of C such that they define a longer prefix of A. Tuple-sets that are defined by i-level exponent sequences are called i-level tuple-sets.

The concept of tuple-sets is a very important concept! It is used constantly throughout this and Schorer's paper. Therefore it is crucial that the reader thoroughly understands this definition. We describe tuple-sets informally again and will give an example afterwards. Remember that for any positive, odd number, there is an infinitely long tuple starting with this number. Now suppose we are given an exponent sequence and we would like to find the tuple-set it defines. Then we would calculate tuples (let them *grow*) as long as their exponent sequence matches the given exponent sequence. As a result, if we were given an i-level exponent sequence A, then no tuple in the tuple-set would contain more than i elements because tuples longer than i elements are excluded by definition (they do not define a *prefix* of A). The fact, that the empty sequence is also a prefix leads to the following: If an exponent sequence of a tuple does not match A at all (i.e. the first elements differ already) then the tuple containing only its first element is contained in the tuple-set. This is because tuples with only one element have an empty exponent sequence and therefore define a prefix of any given exponent sequence.

```
[1]
[3,5]
[5]
[7,11,17,13]
[9]
[11,17]
[13]
[15,23,35]
[17]
[19,29]
[21]
[23,35,53]
[25]
[27,41]
[29]
[31,47,71]
[33]
[35,53]
[37]
[39,59,89,67,101,19]
[41]
[43,65]
[45]
[47,71,107]
[49]
[51,77]
[53]
[55,83,125]
[57]
[59,89]
```

Fig. 1. The first 30 tuples of the tuple-set defined by the 6-level exponent sequence {1,1,2,1,4}. The *shape* is characteristic for this exponent sequence. The first 6-level tuple is starting with 39. It is later defined as the anchor-tuple. It may be a good practice for the reader to understand why every second tuple only consists of only 1 level. The image shows an output of a simple Haskell-program which can be found on our website [?] and in the appendix.

Let us summarize: When we look at any i-level tuple-set, let's say for the i-level exponent sequence A, we will see tuples for every odd number. Tuples can be at most i elements long. An extension of A would only make some tuples longer but no tuple shorter.

When we draw the tuple-set as a list with the odd positive numbers from top to bottom and the tuple elements in each line, we get a characteristic *shape* for every exponent sequence (see Fig. 1).

D. Anchor tuples and marks

A result of the preceding section was, that there is a tuple (at least 1-level) for every odd positive number. This leads directly to the temptation to order the tuples in a tuple-set, namely in the natural order of their first element. We have already seen a picture where this order was applied (Fig. 1). This is convenient since it makes two i-level tuple-sets well comparable. One characteristic we can use to compare two i-level tuple-sets is the first i-level tuple. Schorer calls this tuple the *anchor tuple* of a tuple-set [?, p. 13]. Clearly, the anchor tuple in an i-level tuple-set is unique if it exists. We will show that it exists for every i-level tuple-set later on.

The next term we would like to introduce arises when we ask the other way around: Given an i-level tuple, can we find an i-level tuple-set in which it is the anchor tuple?
Obviously the answer is *No*. For example: Given the tuple $< 11, 17 >$, we can find no 2-level tuple-set in which this is included but not $< 3, 5 >$ (because both define the exponent sequence $\{1\}$).
While the above is not possible, then maybe we can *extend* the exponent sequence in such a way, that the tuple starting with $11, 17$ will eventually become an anchor tuple. That is, letting the i vary. So the question is now: Given a tuple t, can we find some i and some i-level tuple-set in which the i-level prefix of t is the anchor-tuple?
In deed this is possible as we can see with this

Proof: Assume we were given a tuple t which defines an exponent sequence A. Then there is only a finite number n of tuples before it (again assuming the order we used earlier) which define exponent sequences B_1 to B_n. Assuming that all those Bs are different from A, then there must be a least element for every sequence at which they differ first. Therefore, we can find a sufficiently long prefix of A such that all the tuples before t are shorter than t (because their exponent sequences differ from A then) and no other tuple is longer than t (because t defined A). ■

As you might have noticed, we used the assumption that the same exponent sequence cannot be defined by two tuples. The proof to this is given in **??** besides other lemmas.

Let us stick with the last result a little longer. We said, that we only need a sufficiently long prefix of A. This means, that if we would choose a longer prefix, this wouldn't alter the fact that t will be an anchor tuple. This circumstance is described by the term *mark* [?, p. 14].

Let A be an exponent sequence defined by a tuple t. Further, let $A(i)$ be the i-level prefix of A and $t(i)$ the i-level prefix of t. Then the mark *is the element in A with the following attribute: $A(i)$ includes the mark. $\leftrightarrow$ $t(i)$ is the anchor tuple in the i-level tuple-set defined by $A(i)$.*

One implication of this definition is that we can always extend $A(i)$ such that the $i-level$ the prefix of t stays the anchor tuple (namely the extension is $A(i+1)$). Informally: We can carefully extend any exponent sequence such that the anchor tuple in the tuple-set it defines does not change.

We know enough terminology now to look at some lemmas and then at the actual proof.

But before we do that, we would like to give a very short summarization of what we have discussed so far.

E. Summarization

For clarity we present the most important facts in a tabular view:

1) Every odd, positive integer defines an infinite tuple.
2) Every tuple defines an exponent sequence.
3) Every i-level exponent sequence defines an i-level tuple-set.
4) Every tuple-set has an anchor tuple.
5) Every exponent sequence has a mark.

III. Lemmas and Proof

Most of the proofs you will see here have been altered from the their appearance in Schorer's paper but the ideas remain his. Any flaws may well be ours. We will cover the lemmas first and move then on to the actual proof.

A. Lemmas

In his paper, Schorer gives more than a dozen of lemmas, not all of which are actually used in the proof. Because of the limits of this paper, we will only give a selection of the most important lemmas here. For every lemma, we will give a reference to the according lemma in Schorer's paper. First, we prove subsequently the lemma that we already assumed in the previous section.

Lemma 1 : Different tuples define different exponent sequences. (no according lemma in Schorer's paper)

Proof: (by contradiction) Assume two tuples are different and they share the same exponent sequence. Since the tuples are different, there exists a first elements at which they differ, say x and y. We assume w.l.o.g. $y \geq x+2$ (this is possible because they only contain odd, positive integers).
Since the exponent is the same for both we know $C(y) \geq C(x+2) \geq C(x)+2$. From this, we see that the difference between their tuple values grows by at least two with each step (i.e., application of C). This however tells us that after sufficient steps, $C(y) \geq 2 \cdot C(x)$. But then the exponent for the next application on y would have to be at least greater by one than the exponent for x. I.e. the exponent sequences differ. ∎

Lemma 2 : If one counterexample exists, then infinitely many counterexamples exist. [?, similar to Lemma 11, p. 31]

Proof: We call the counterexample given in this lemma x and say $C(x) = y$. Note that y is a counterexample as well (not necessarily different from x). We show a way to construct another odd integer x' that too, maps directly to y and is therefore also a counterexample. And because x' will be strictly greater it is also different from x. Then the construction formula can be applied to x' to generate yet another counterexample and so on. Specifically:
If

$$C(x) = \frac{3x+1}{2^a} = y$$

then $C(x+2^a y)$ equals also y. That this is true can be seen by simple algebra:

$$C(x+2^a y) = \frac{3(x+2^a y)+1}{2^a \cdot 4}$$

by plugging in the previous condition, we have

$$\frac{3(x+(3x+1))+1}{2^a \cdot 4} = \frac{4(3x+1)}{2^a \cdot 4} = y$$

Finally, because x is a positive and odd integer, $x' = x + 2^a y$ is. ∎

Lemma 3 : For every i-level exponent sequence, there exists an i-level tuple. [?, similar to Lemma 13, p. 34]

Proof: If $i = 2$ then the exponent sequence only contains one element a_1. So to fulfill the lemma, we need to find two odd numbers $x, y \in \mathbb{N}$ such that $C(x) = y$. When the exponent sequence gets longer, we need to fulfill $C(\dots C(x)\dots) = y$. The formula for n applications of C is

$$\frac{3^n x + r}{2^{a_1+\cdots+a_n}}$$

This formula and the fact that r is always odd is not deduced here but can easily be proved correct by writing down the first few iterations or can be found in [?, p. 36]. For the next steps it's more convenient to write the relation as

$$x = \frac{y \cdot 2^{a_1+\cdots+a_n} - r}{3^n} \tag{3}$$

To see, that x is an integer, we need to show that $y \cdot 2^{a_1+\cdots+a_n} - r \equiv 0 \mod 3^n$. In Number Theory class we learn that this is true when $gcd(2^{a_1+\cdots+a_n}, 3^n) = 1$ which of course is true because they are factorized by different primes. The general solution then to (**??**) is $y = y_0 + k3^n$ for a particular solution y_0 and any $k \in \mathbb{Z}$. So this way we can always find a y which is odd and positive and we also see that x is odd and positive then by looking at (**??**). ■

Lemma 4 : For every i-level exponent sequence with the last element variable, there exists an i-level tuple that ends in a fixed element. [?, Lemma 14, p. 36]

Proof: You might have noticed the similarity to the previous lemma. There, we have seen that we can find an i-level tuple for a given exponent sequence. This time we require this tuple to end in a specific number. To achieve this we are given the choice over the last element in the exponent sequence. The proof is similar to the one above. In (**??**) we have y as a constant now and a_n as the variable. With this in mind look at the congruence again: $(y \cdot 2^{a_1+\cdots+a_{n-1}})2^{a_n} - r \equiv 0 \mod 3^n$. It is $gcd(y2^{a_1+\cdots+a_{n-1}}, 3^n) = 1$ because y is not divisible by 3 (this is because it can be written as $3x + 1/2^a$ and would leave a rest 1 for every division by 3). Further $gcd(2^{a_n}, 3^n) = 1$ as well as $gcd(r, 3^n)$ (again, to see the structure of r we refer to [?, p. 37]). So they are all invertible which means that both sides of the congruence can be inverted and the congruence solved. ■

Lemma 5 : Assume a counterexample exists. Then in every i-level tuple-set, there is an infinity of i-level counterexamples and an infinity of i-level non-counterexamples. [?, Lemma 5, p. 28]

Proof: We are given an i-level exponent sequence and a counterexample. From Lemma 2 we know, that if one counterexample exists, infinitely many counterexamples exist. From Lemma 1 we know that there exists at least one i-level tuple in the defined i-level tuple-set. And eventually from Lemma 4 we know, that we can extend the given i-level exponent sequence A to an $i + 1$-level exponent sequence A', such that a tuple t that defines e' (again, we know it exists because of Lemma 1) includes a given number at position $i + 1$. In our case of course, we pick a number that lies in one of the infinite counterexample tuples. Then t becomes a counterexample itself and it defines also the i-level exponent sequence we wanted. We can do this for any counterexample to get infinitely tuples with the above features. Also their i-level prefix is different because at least their elements at position i are different. This is because their elements at position $i + 1$ differ by design.
Similarly this applies to non-counterexamples. ■

B. The Proof

The idea of the following proof was found in Schorer's paper. However, it has been altered by us in order to make it better understandable in our sense. Building up on this idea, some similar proofs can be found. In fact, Schorer gives three proofs in his paper. This one would most likely correspond to his second proof [?, pp. 17].

Proof: Seeking a contradiction, we assume that a counterexample exists. Remember that every odd positive integer is either a counterexample or a non-counterexample which means that they are different and therefore define different tuples. From Lemma 1 we know that they therefore also define different exponent sequences. So we know that the exponent sequence defined by the counterexample tuple is different from any exponent sequence of a non-counterexample tuple.
That they are different means that there is a first element at which they differ. Then however, the tuple-set defined by a prefix long enough to include this first element, would not contain any i-level non-counterexample tuple. This is because otherwise this non-counterexample tuple then would in turn define an exponent sequence which has the prefix we said cannot exist in the set of non-counterexample exponent sequences.
This is a contradiction to Lemma 5 which states, that every i-level tuple-set contains infinitely many non-counterexamples.
Therefore the assumption is wrong and no counterexample-tuples and therefore no counterexamples exist. ■

Crosscheck for the fact that the counterexample exponent sequence cannot be included in the set of non-counterexample exponent-sequences: If it was, this exponent sequence would have a mark and a tuple-set defined by a prefix of this exponent sequence including the mark would then have a counterexample anchor tuple and a non-counterexample anchor tuple. But there can only be one anchor tuple by definition.

Crosscheck for the fact that the counterexample exponent sequence must be included in the set of non-counterexample exponent-sequences: From Lemma 4 we know, that every prefix of an exponent sequence that is associated with a counterexample tuple can be extended such that it is associated with a non-counterexample tuple (and vice versa). So every prefix of this counterexample exponent sequence must be included in the set of non-counterexample exponent sequences.

IV. Criticism and Comments

Writing about Schorer's paper, one cannot omit the critical voices that come with it. Although we have not found a flaw in any of Schorer's proofs, we feel there are several areas of concern. As a consequence of the fact that it has not been disproved we can only give general arguments and conjectures in this section.

A. Lack of reputation

According to himself, Peter Schorer is a former researcher at Hewlett-Packard's main research laboratory - Hewlett-Packard Labs in Palo Alto, Calif. His degree is in computer science. Although the paper we are referring to is relatively new, there were already earlier versions in 2008. In his paper he writes that the effort he has put into informing magazines and journals to publish his proof has been without success [?, Appendix C].
The reason he assumes, is his lack of reputation. Meaning that since he is not a professional mathematician, journal editors and their referees are skeptical that a proof to such an old mathematical problem can come from someone outside the mathematical community.

B. Lack of formality

Besides the reputation of the author, we believe that a certain lack of formalism throughout his work feeds skepticism as well (see

next section ??). When complicated circumstances are explained with words only, for example, mathematical cases can be overlooked. Especially when it comes to infinities, things are often not intuitive and his paper deals with infinities at many places.
When it comes to the actual proof, everything is explained by good thinking (except the referenced lemmas). Some of the steps he takes seem to be too wide to be proven complete and correct easily.

C. Possible Problems

While dealing with this proof, we noticed some possible sources of errors which we will share here. These are points that can be seen as origins for further investigations.

1. Infinity. Dealing with the infinities of tuples, tuple-sets and exponent sequences can be error prone. The human mind tends to think of them as being finite which might lead to false assumptions.

2. Semi-Decidability. This point applies to the proof presented in this paper. It is only semi-decidable whether a number is a counterexample or not and therefore picking a counterexample and setting down the rest of the proof with it might lead to formal problems.

3. Even numbers. Although we said, that even numbers are reduced to odd numbers, the relation between them and the odd numbers they are reduced to, has not been discussed. In any case, it would be helpful to demur and (shortly) mention even numbers when proofing the conjecture and certain lemmas.

4. Too good to be true. Assuming, that the proof is correct, it is surprising that the evidence is so manifold. By this we think of the following fact. In the proof presented here, we only need one counterexample. However, Lemma 2 shows us, that if there exists one counterexample, then there are infinitely many. By our own experience this excess of evidence would mean that the proven fact is somewhat obvious or that easier proofs exist. But both can hardly be true if the problem has been analyzed for more than 70 years.

D. Shared authorship

One last remark for the interested reader. On his homepage [?], Peter Schorer offers shared authorship for anyone who can improve the paper so that it leads to direct publication. One way we see is to bring the idea of the proof down to a formal level so that reasoning is possible in the language of mathematics. In other words, until now, the Collatz-Problem has been reduced to proving a claimed proof.

APPENDIX
GENERATING TUPLE-SETS WITH HASKELL

We present a short program that can be used to generate tuple-sets of arbitrary size for a given exponent sequence. It has been written in Haskell and therefore requires a Haskell interpreter (e.g., Hugs http://cvs.haskell.org/Hugs/). The program can also be obtained online at [?]. To print an example tuple-set for the exponent-sequence $\{1,1,2\}$ type
`printts(take10(tupleset[1,1,2]))`
at the prompt.

File: **C.hs**

```
1 c x a = if (q == 0 && odd p) then p else 0
2     where (p,q) = divMod (3 * x + 1) (2 ^ a)
3 tuple eseq x = takeWhile (/= 0) (scanl c x eseq)
4 tupleset eseq = map (tuple eseq) [1,3..]
5 printts xs = mapM˙ print xs
```

Line 1 (and 2): c x a returns the next tuple element calculated as $(3x + 1)/(2^a)$

if the result is an odd positive integer or 0 otherwise.

Line 3: `tuple` $eseq$ x returns the tuple starting with x that defines the longest possible prefix of the exponent sequence $eseq$.

Line 4: `ttupleset` xs returns an infinite tupleset defined by the exponent sequence $eseq$.

Line 5: `tprintts` is used for pretty-printing the output (one tuple per line).

REFERENCES

[1] P. Schorer. (2009, Feb.) A solution to the 3x+1 problem. 2538 Milvia St., Berkeley, CA. [Online]. Available: http://www.occampress.com/#3x+1

[2] J. C. Lagarias, "3x+1 problem annotated bibliography," Murray Hill, New Jersey, Feb. 1996.

[3] ——, "The 3x + 1 problem and its generalizations," *The American Mathematical Monthly*, vol. Vol. 92, No. 1, pp. 3–23, Jan. 1985. [Online]. Available: http://www.cecm.sfu.ca/organics/papers/lagarias/paper/html/paper.html

[4] G. J. Wirsching, "A functional differential equation and 3n+1 dynamics," in *Topics in Functional Differential and Difference Equations*. AMS Bookstore, 2000, pp. 360 – 378. [Online]. Available: http://books.google.com/books?id=Yz6GFA-jtgwC

[5] ——, "The dynamical system generated by the 3n+1 function," in *Lecture Notes in Mathematics*. Springer Berlin / Heidelberg, Nov. 2006, vol. Vol. 2. [Online]. Available: http://www.springer.com/math/numbers/book/978-3-540-63970-1

[6] E. Roosendaal, "On the 3x+1 problem." [Online]. Available: http://ericr.nl/wondrous/

[7] R. Haynberg, "State of the 3x+1 problem," Mar. 2009. [Online]. Available: http://rolf.haynberg.de/?p=429